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Practice Midterm 2 December 3, 2008

1. $\mathbb{N}$ is the set of natural numbers $1, 2, 3, \ldots$. $A \subseteq \mathbb{N}$ belongs to a collection $\mathcal{T}$ if $A = \emptyset$ or $\sup\{n \ : \ n \notin A\} < \infty$. Show that $\mathcal{T}$ defines a topology on $\mathbb{N}$. Is this topology Hausdorff?

   Solution. (i) $A = \emptyset \in \mathcal{T}$, and the sup over all n such that $n \notin A$ is finite, so $A = \mathbb{N} \in \mathcal{T}$.

   (ii) Let $U, V \in \mathcal{T}$.

   case 1: $U = \emptyset$ V is any set $\neq \emptyset$. $U \cap V = \emptyset \in \mathcal{T}$

   case 2: both $U, V = \emptyset$. This has same result as above

   case 3: $U, V \in \mathcal{T}$, $x_U = \sup\{n : n \notin U\}$ and $x_V = \sup\{n : n \notin V\}$. Both $x_U, x_V < \infty$. Notice that $\max(x_U, x_V) < \infty$. Therefore, $x_{U\cap V} = \max(x_U, x_V) = \sup\{n : n \notin U \cap V\} < \infty$. Therefore, $U \cap V \in \mathcal{T}$.

   (iii) Let $\mathcal{U} \subset \mathcal{T}$ and let $U \in \mathcal{U}$. So, either $U = \emptyset$ or $\sup\{n : n \notin U\} < \infty$. The first case is not very interesting, if all such $U = \emptyset$ then the union will be the emptyset and this will be in $\mathcal{T}$. Now, define $x_{\text{sup}} = \sup\{n : n \notin \bigcup_{U\in\mathcal{U}} U\} = \inf_U \sup\{n : n \notin U\} < \infty$. So, $\bigcup_{U\in\mathcal{U}} U \in \mathcal{T}$.

   $\therefore \mathcal{T}$ is a topology on $\mathbb{N}$.

   Is $\mathcal{T}$ Hausdorff? I claim that it isn't. All sets that are not the emptyset, must have a form similar to

   $$\begin{aligned} \{m, m+1, \ldots\} &= U \\ \{n, n+1, \ldots\} &= V \end{aligned}$$

   where $m, n \in \mathbb{N}, m \neq n$ and WLOG $m > n$. Sets must be of this form since the sups of elements not in the sets are finite. Sets of this type will have non-empty intersection, so the $\mathcal{T}$ is not Hausdorff.

2. A topology on the space of smooth functions on $\mathbb{R}$ is defined by the sets

   $$U(\phi, m.n.\epsilon) = \left\{\psi \text{ is a smooth function}, \left|\frac{d^k(\phi - \psi)}{dx^k}\right| < \epsilon, \ \forall |x| < n, 0 \leq k \leq m\right\}.$$

   Define a map from smooth functions to $\mathbb{R}^2$ by $\phi \mapsto \Big(\phi(0) - \phi(1), \int_0^1 [\phi'(x)]^2 dx\Big)$. Show that this map is continuous.

   Solution. Call the map g. Showing that g is continuous amounts to showing that the each component of the mapping of g to $\mathbb{R}^2$ is continuous. For i^{th} $(i = 1, 2)$ component, continuity means: given $\epsilon > 0$ $\exists U(\phi; m_\epsilon, n_\epsilon, \delta_\epsilon)$ s.t. $\psi \in U \implies |g(\phi) - g(\psi)|_i < \epsilon$.

For the first component, pick $m_\epsilon = 0$, and $n_\epsilon = 2$. We have

$$\begin{aligned} |g(\phi) - g(\psi)|_1 &= |(\phi(0) - \phi(1)) - (\psi(0) - \psi(1))| \\ &\leq |\phi(0) - \psi(0)| + |\phi(1) - \psi(1)|. \end{aligned}$$

Now, I control the difference between any ψ and ϕ in U by the choice of $n_\epsilon = 2$. Therefore, if I choose $\delta_\epsilon = \epsilon/2$, then

$$|g(\phi) - g(\psi)|_1 < \epsilon/2 + \epsilon/2 = \epsilon$$

so the first component is continuous.

For the second component, pick $m_\epsilon = 1$, and $n_\epsilon = 2$. We have

$$\begin{aligned} |g(\phi) - g(\psi)|_2 &= |\int_0^1 (\phi'(x)^2 - \psi'(x)^2)dx| \\ &\leq \int_0^1 |(\phi'(x)^2 - \psi'(x)^2)|dx \\ &= \int_0^1 |\phi' + \psi'||\phi' - \psi'|dx. \end{aligned}$$

Bound, through simple choice, ψ' by $\phi' + 1$. Let $M = \sup_{0 \leq x \leq 1} |\phi'(x)|$. Now I can bound the first absolute value in the integral by

$$|\phi' + \psi'| \leq |M + (M + 1)| = |2M + 1|.$$

So, now

$$|g(\phi) - g(\psi)|_2 \leq |2M + 1| \int_0^1 |\phi' - \psi'|dx < |2M + 1|\delta_\epsilon$$

So, by construction, choose $\delta_\epsilon = \min(1, \frac{\epsilon}{|2M+1|})$. This makes the second component of the map g continuous. $\therefore g$ is continuous.

3. The notes define what it means for two bases to be equivalent. Write down the negation of this statement, *i.e.* define what it means for two bases to *not be equivalent*. Using this or otherwise, show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two first countable topologies on X, such that $x_n \to x$ in $\mathcal{T}_1$ iff it also converges in $\mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$. (Hint: Identity map!).

   Solution.

   Definition. Two bases $\mathcal{B}_1, \mathcal{B}_2$ are not equivalent:

   if $\exists A_1 \in \mathcal{B}_1$ with $x \in A_1$ s.t. $\forall A_2 \in \mathcal{B}_2$ either $x \notin A_2$ or $A_2 \not\subset A_1$.

   or

   if $\exists A_2 \in \mathcal{B}_2$ with $x \in A_2$ s.t. $\forall A_1 \in \mathcal{B}_1$ either $x \notin A_1$ or $A_1 \not\subset A_2$.

   ($\Longleftarrow$) Assume $\mathcal{T}_1 = \mathcal{T}_2$. Then if $x_n \to x$ in T_1, it must also converge to x in $\mathcal{T}_2$, and if $x_n \to x$ in $\mathcal{T}_2$, then it must also converge to x in $\mathcal{T}_1$.

($\Longrightarrow$) Assume $x_n \to x$ in $\mathcal{T}_1$ iff $x_n \to x$ in $\mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are first countable, sequential continuity implies continuity.

Take the identity map between $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$. This mapping is sequentially continuous. So, every open set in $\mathcal{T}_2$ gets pulled back to an open set in $\mathcal{T}_1$. So, $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

The same argument follows for $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Therefore, $\mathcal{T}_1 = \mathcal{T}_2$.

$\therefore$ if $x_n \to x$ in $\mathcal{T}_1$ iff it also converges in $\mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

4. Prove or disprove: $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous. $\mathcal{T}$ is second countable. Then $\mathcal{S}$ is also second countable.

   Solution. This statement is false.

   Consider the identity map from $(l^2, \mathcal{T}_{\text{metric}}) \to (l^2, \mathcal{T}_{\text{weak}})$. $\mathcal{T}_{\text{metric}}$ is second countable, however since not every open set in $\mathcal{T}_{\text{metric}}$ is open in $\mathcal{T}_{\text{weak}}$, the base for $\mathcal{T}_{\text{weak}}$ will not be second countable.

5. If $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$, show that, for each index i, $x_i^{(n)} \to x_i$. Using this or otherwise, show that (strongly) closed unit l^2 ball $\{\mathbf{x} | ||\mathbf{x}||_2 \leq 1\}$ is also weakly closed. (Hint: Consider the functions $f_k(\mathbf{x}) = \sum_{i=1}^k x_i^2$. Note also that the (strongly) open unit ball $\{\mathbf{x} | ||\mathbf{x}||_2 < 1\}$ is *not* weakly open!)

   Solution.